Why Einstein, Podolsky and Rosen did not prove that quantum mechanics is 'incomplete'

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Abstract

It is shown that the Einstein-Podolsky-Rosen conclusion concerning the 'incompleteness' of Quantum Mechanics is invalidated by two logical errors in their argument. If it were possible to perform the proposed gedanken experiment it would, in fact, show that Quantum Mechanics is 'complete' for the observables discussed. Because, however, of the non square-integrable nature of the wave function, the proposed experiment gives vanishing probabilities for measurements performed in finite intervals of configuration or momentum space. Hence no conclusion as to the 'completeness', or otherwise, of Quantum Mechanics can be drawn from the experiment.

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Perhaps no other paper written in the 20th Century generated as much debate about questions related to the foundations of physics and their philosophical implications than that of Einstein, Podolsky and Rosen (EPR) [1]. However, after the initial replies written by Bohr [2], Furry [3] and Schrödinger [4], there has been very little critical discussion of the EPR paper itself in the literature[a]. In this article a reappraisal of the EPR paper is made and the following conclusions are drawn:

(i) The argument presented by EPR to demonstrate the 'incompleteness' of Quantum Mechanics (QM) is invalidated by two logical errors.

(ii) The gedanken experiment proposed by EPR cannot be carried out if the usual probabilistic interpretation of QM is correct, and so no physical conclusions can be drawn from the experiment.

Following EPR, a theory is said to give a 'complete' description of a physical quantity if the following condition is satisfied:

'Without, in any way, disturbing a system, we can predict with certainty (i.e. with probability equal to unity) the value of the physical quantity'.

If this is the case, EPR associate an 'Element of Physical Reality' to the the corresponding quantity. EPR also require that, in a 'complete' theory:

'Every Element of Physical Reality must have a counterpart in the physical theory'.

This hypothesis is not particularly important since it must necessarily be true if the theory is able to predict the value of the corresponding physical quantity.

The EPR gedanken experiment will first be discussed from a purely logical viewpoint. Secondly, the conceptual feasiblity, within QM, of the proposed experiment is examined. The following hypotheses are defined:

- QMT : QM is a true theory within its domain of applicablity.
- $QMTC(A, B, ..)$: QM is a true, complete, theory for the physical quantities A,B,.. .
- $PRNC(A, B)$: Elements of Physical Reality exist for each of a pair of physical quantities A, B with non-commuting operators in QM.

The EPR gedanken experiment is based solely on the hypothesis QMT (Quantum Mechanics True). Contrary to the statement of EPR, it is *not necessary* to assume, at the outset, that QM is also a complete theory (hypothesis $QMTC$). This is EPR's first logical error. In fact, applying QMT and assuming also that a quantum mechanical system of two correlated particles with a certain well-defined wave function can be constructed, EPR found that Elements of Physical Reality apparently *can* be assigned to each of the quantities P and Q that that have non-commuting operators. EPR thus found that the proposition $PRNC(P, Q)$ follows from QMT alone according to their interpretation of the results of the gedanken experiment. After correction[b], the final statement of the result of the gedanken experiment is:

[a] Almost all subsequent discussion of 'EPR experiments' in the literature is, instead, based on Bohm's gedanken experiment [5] involving correlated spin measurements.

[b] i.e. replacing in the statement of EPR the hypothesis $QMTC$ by QMT.

'Starting from the assumption of the correctness of QM (i.e. hypothesis QMT) we arrived at the conclusion that two physical quantities with non-commuting observables can have simultaneous reality.'

According to EPR's definitions, if two physical quantities have corresponding elements of physical reality, then the theory is a complete one for these quantities. In symbols[c]:

$$PRNC(P,Q) \otimes QMTC(P,Q) = TRUE \tag{1}$$

Using De Morgan's Theorem, (1) implies:

$$\overline{PRNC(P,Q)} \oplus \overline{QMTC(P,Q)} = FALSE \tag{2}$$

where in (2) and the following the symbol '$\oplus$' denotes an 'exclusive or' [d]. However, as will be seen below, EPR state that the right side of (2) is TRUE from which it follows instead of (1), that:

$$PRNC(P,Q) \otimes QMTC(P,Q) = FALSE \tag{3}$$

Since the gedanken experiment shows that:

$$PRNC(P,Q) = TRUE,$$

the erroneous conclusion is drawn, on the basis of (3), that:

$$QMTC(P,Q) = FALSE.$$

i.e. that QM is an incomplete theory. This was EPR's final conclusion. It was actually reached using a *reductio ad absurdum* argument that is reviewed below. The basic assertion of EPR (actually, as shown above, in contradiction to the result of their gedanken experiment) is the negation of the proposition (2):

$$\overline{PRNC(P,Q)} \oplus \overline{QMTC(P,Q)} = TRUE \tag{4}$$

This is EPR's second logical error. How is this assertion justified in the EPR paper? After discussion of quantum mechanical measurements on a *single particle*, with no obvious relevance to the case of *two correlated particles* as used in their gedanken experiment, EPR state that:

'From this it follows [1]*the quantum mechanical description of reality given by the wave function is not complete or* [2]*when the operators corresponding to two physical quantities do not commute the two quantities cannot have simultaneous reality.* For if both of them had simultaneous reality - and thus definite values - these values would enter into the complete description according to the condition of completeness. If the wave function provided such

[c]Each hypothesis is assumed to be either true or false. The symbols $\otimes$ and $\oplus$ denote, respectively logical 'and' and 'exclusive or'. The latter is defined in such a way that if $X \oplus Y = TRUE$. the possiblities that X and Y are both $TRUE$, or both $FALSE$, are excluded. A bar on a logical variable indicates negation.

[d]The meaning of the 'exclusive or' $X \oplus Y = FALSE$ is that either X and Y are both true, or they are both false.

a complete description of reality it would contain these values, these would be predictable. This not being the case we are left with the alternatives stated.'

(Italics in the original)

The italicised statement, expressed symbolically by Eqn.(4), seems to be justified by the preceding discussion in the paper of non commuting observables (position and momentum) for a *single* particle, not the correlated two particle system of the gedanken experiment subsequently presented. In fact no justification is given by EPR for the application, *a priori*, of propositions [1] and [2] and their relation Eqn.(4) to the gedanken experiment. Even so, one can still ask what is the meaning of EPR's assertion in Eqn.(4)? As quoted above, EPR carefully explain that the negation of the proposition [2] implies that the quantum mechanical description of two commuting observables is complete, i.e.

$$\overline{[2]} \equiv PRNC(A,B) \Rightarrow QMTC(A,B)$$

or, equivalently,

$$[2] \equiv \overline{PRNC(A,B)} \Rightarrow \overline{QMTC(A,B)}$$

where the symbol $\Rightarrow$ is used for 'logically implies', i.e. $X \Rightarrow Y$ means that if X is $TRUE(FALSE)$ then Y is $TRUE(FALSE)$ and *vice versa*. But just this condition i.e. both $\overline{PRNC(A,B)} = TRUE$ and $\overline{QMTC(A,B)} = TRUE$ is one of the two possibilities that are excluded by the definition of the 'exclusive or' proposition (4) that EPR assume to be correct!

To summarise, the EPR argument is based on the 'exclusive or' proposition:

$$X \oplus Y = TRUE$$

which implies that the only possiblities are: $X = FALSE$ and $Y = TRUE$ or *vice versa*, the cases when X and Y are both true or both false being excluded. But the truth (or falsehood) of the proposition $X \equiv \overline{PRNC(P,Q)}$ entails the truth (or falsehood) of the proposition $Y \equiv \overline{QMTC(P,Q)}$. Therefore if X is false —the claimed conclusion of EPR's analysis of their gedanken experiment— Y must also be false, so that QM is then a complete theory, not an incomplete one as claimed by EPR. In fact the true conclusion, that both X and Y are false, contradicts EPRs initial proposition (4). As pointed out above, when X and Y are both false the '$TRUE$' on the right side of EPR's initial proposition must be replaced by '$FALSE$'. Indeed, $\overline{PRNC(P,Q)}$ —quantum mechanics is incomplete for the non-commuting variables P,Q— is a special case implied by the more general proposition $\overline{QMTC(A,B)}$ for arbitary non-commuting variables A and B, when $A = P$ and $B = Q$.

An example of an absurd (self-contradictory) conclusion that is obtained from asserting the truth of an exclusive or proposition, when both of the related propositions is true, is the following. Assume such a proposition is true when X = 'Aristotle was mortal' and Y = 'Aristotle was a man'. Men are a subset of all mortal beings just as the 'incomplete' variables in $\overline{PRNC(P,Q)}$ are a subset of those in $\overline{QMTC(A,B)}$. Since Aristotle died, X is true. It follows then from the 'exclusive or' that Aristotle was not a man, in contradiction to the inital proposition Y!

Since EPR were assuming the correctness of the 'exclusive or' (4) and the gedanken experiment is claimed to show that[e] $PRNC = TRUE$ or $\overline{PRNC} = FALSE$, they could have immediately concluded from (4) that $\overline{QMTC} = TRUE$ —quantum mechanics is not complete. Actually, however, they reached the same conclusion, based on (4), by a more convoluted *reductio ad absurdum* argument. EPR introduced, as well as (4), the initial hypothesis: $QMTC = TRUE$. They then claimed that the result of the gedanken experiment which was $\overline{PRNC} = FALSE$ and hence also $\overline{QMTC} = FALSE$ was a logical consequence of the assumption $QMTC = TRUE$. In spite of the fact that $\overline{QMTC} = FALSE$ is logically equivalent to the claimed initial proposition $QMTC = TRUE$, EPR noted that the result of the gedanken experiment, $\overline{PRNC} = FALSE$, together with the proposition (4), the correctness of which they assume, implies that $\overline{QMTC} = TRUE$, in contradiction with their initial proposition $QMTC = TRUE$. EPR then deduce from this contradiction, by *reductio ad absurdum*, that the initial propostion $QMTC = TRUE$ must be false, so that quantum mechanics is incomplete. In fact the result of the gedanken experiment is consistent with the assumption $QMTC = TRUE$, it is the other initial proposition,(4), that must be rejected as erroneous. Indeed the proposition (4) is erroneous —self contradictory— given only the meanings of the two related propositions. The *reductio* argument is therefore valid, but the intial hypothesis that must be rejected is not $QMTC = TRUE$ but (4)! As stated above in any case the initial hypothesis of the gedanken experiment is QMT *not* $QMTC$ and the result $QMTC = TRUE$ is derived from the gedanken experiment without, contrary to EPR's assertion, first assuming that $QMTC = TRUE$.

Correcting the logical errors described above, it might seem that the EPR experiment establishes the 'completeness' of quantum mechanics for the two non-commuting quantities P and Q. For this, however, it is necessary that the suggested gedanken experiment can, at least in principle, be performed. It will now be shown that this is not the case, so that no conclusion can be drawn as to the 'completeness', or otherwise, of quantum mechanics, by the arguments presented by EPR.

The spatial wave function of the correlated two particle system discussed by EPR is:

$$\Psi(x_1, x_2) = \int_{-\infty}^{\infty} dp \exp \frac{2\pi i}{h}(x_1 - x_2 + x_0)p = h\delta(x_1 - x_2 + x_0) \qquad (5)$$

The probability that the particle 1 will be observed in the interval $a < x_1 < b$, for any position of the particle 2, can be written as:

$$\begin{aligned} P(a < x_1 < b) &= Lim(L \to \infty) \frac{\int_a^b dx_1 \int_{-\infty}^{\infty} dx_2 |\Psi(x_1, x_2)|^2}{\int_{-L}^{L} dx_1 \int_{-\infty}^{\infty} dx_2 |\Psi(x_1, x_2)|^2} \\ &= Lim(L \to \infty) \frac{b - a}{2L} = 0 \end{aligned} \qquad (6)$$

[e] Since the arguments, P and Q, of $PRNC$ and $QMTC$ are the same they are omitted, for brevity, in the following

The particle 1 cannot, therefore, be observed in any finite interval of x_1, and so the Q measurement suggested in the EPR gedanken experiment cannot be carried out.

By making Fourier transforms with respect to x_1 and x_2 the momentum wavefunction corresponding to (5) is found to be:

$$\Psi(p_1,p_2) = \frac{h^2}{2\pi} \exp \frac{2\pi i p_1 x_0}{h} \delta(p_1+p_2) \tag{7}$$

The probability to observe p_1 in the range $p_a < p_1 < p_b$ for any value of p_2 is:

$$\begin{aligned} P(p_a < x_1 < p_b) &= Lim(p\to\infty)\frac{\int_{p_a}^{p_b} dp_1 \int_{-\infty}^{\infty} dp_2 |\Psi(p_1,p_2)|^2}{\int_{-p}^{p} dp_1 \int_{-\infty}^{\infty} dp_2 |\Psi(p_1,p_2)|^2} \\ &= Lim(p\to\infty)\frac{p_b-p_a}{2p} = 0 \end{aligned} \tag{8}$$

The momentum of particle 1 cannot be measured in any finite interval so that the proposed $p_2 = P = -p_1$ measurement of the EPR gedanken experiment cannot be carried out. In fact, the correlated two particle wave function proposed by EPR is not square integrable either in configuration or momentum space and so has no probabilistic interpretation in QM. The single particle wavefunction discussed by EPR has the same shortcoming. Hence the 'relative probability' $P(a,b)$ of EPR's Equation (6) also vanishes. While the statement that 'all values of the coordinate are equally probable' is true, it is also true that the absolute probability to observe the particle in any finite coordinate interval is zero.

The EPR two particle wavefunction is now modified to render it square integrable so that the results of the gedanken experiment may be interpreted according to the usual rules of QM. The suggested 'minimally modified' wavefunction is:

$$\tilde{\Psi}(x_1,x_2) = \frac{1}{(\sqrt{2\pi}\sigma_x)^{\frac{1}{2}}} \exp\left(\frac{x_0^2 - 2x_1^2 - 2x_2^2}{16\sigma_x^2}\right) \delta(x_1 - x_2 + x_0) \tag{9}$$

Like the EPR wavefunction (5), $\tilde{\Psi}$ vanishes unless $x_2 = x_1 + x_0$, but it is square integrable and normalised:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\tilde{\Psi}(x_1,x_2)|^2 dx_1 dx_2 = 1 \tag{10}$$

The EPR wavefunction (5) is recovered in the limit $\sigma_x \to \infty$. Performing a double Fourier transform on Eqn(9) yields the corresponding momentum wave function:

$$\tilde{\Psi}(p_1,p_2) = \frac{1}{\pi\sigma_p} \exp\left(-\frac{(p_1+p_2)^2}{2\sigma_p^2}\right) \exp\left(\frac{2\pi i p_1 x_0}{h}\right) \tag{11}$$

where

$$\sigma_p = h/4\pi\sigma_x$$

The wavefunction (11) is also square integrable and normalised:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\tilde{\Psi}(p_1,p_2)|^2 dp_1 dp_2 = 1 \tag{12}$$

and the EPR wavefunction (7) is recovered in the limit $\sigma_x \to \infty$, $\sigma_p \to 0$. Now, performing the EPR gedanken experiment, using instead the wavefunctions (9) and (11) it becomes clear that it is no longer possible to associate 'Elements of Physical Reality' to the position Q and the momentum P of the second particle by performing measurements on the first one. The probability $\delta P(x_1)$ that the spatial position of the first particle lies in the interval δx_1 around x_1[f] is:

$$\delta P(x_1) = \frac{1}{\sqrt{\pi}\sigma_x}\exp\left(-\frac{x_0^2}{8\sigma_x^2}\right)\exp\left(-\frac{(2x_1+x_0)^2}{8\sigma_x^2}\right)\delta x_1 \tag{13}$$

Because of the δ-function in the wave function (9), this is also the probability that x_2 lies in the interval of width $\delta x_2 = \delta x_1$ around $x_2 = x_1 + x_0$. Measuring x_1 in the interval δx_1 then enables the certain prediction that x_2 lies in the interval δx_2 around $x_2 = x_1 + x_0$. However, to associate an 'Element of Physical Reality' to x_2 requires that the *value* must be exactly predictable. For this it is necessary that $\delta x_1 = \delta x_2 \to 0$. In this case $\delta P(x_1)$ vanishes and no possibility exists to measure the position of the particle 1. The situation is then the same as in the case of the original EPR wavefunction (5). It is clear that, using the wavefunctions (9) and (10), the product of the uncertainties in P and Q can be much smaller than that required by the Heisenberg uncertainty principle. However in order to thus determine Q, use is made of of the precise knowledge of the parameter x_0 of the wavefunction, i.e. exact knowledge of how the wavefunction is prepared is required. But if *a priori* knowledge about wavefunction preparation is admitted, it is trivial to show that observables with non-commuting operators can be simultaneously 'known' with a joint precision far exceeding that allowed by the momentum-space uncertainty relation. To give a concrete example of this, the process of para-positronium annihilation at rest: $e^+e^- \to \gamma\gamma$ may be considered. The uncertainty in the momentum Δp of one of the decay photons is determined by the energy-time uncertainty relation[g]and the kinematical relation $c\Delta p = \Delta E$ to be:

$$\Delta p = \frac{h}{c\tau}$$

where the mean lifetime of the decay process $\tau = 1.25 \times 10^{-10}$ sec. The momentum-space uncertainty relation then predicts

$$\Delta x > 3.75\text{cm}$$

The technically simple measurement of the position of the photon in the direction parallel to its momentum to within ± 1mm (for example, by observing

[f] i.e. that x_1 lies between $x_1 - \delta x_1/2$ and $x_1 + \delta x_1/2$

[g] This is derived from the Fourier transform of the exponential decay law which gives a Breit-Wigner function of width $\simeq c\Delta p$.

a recoil electron from Compton scattering of the photon [6]) then allows simultaneous knowledge of the position and momentum of the photon (whose quantum mechanical operators do not commute) with an accuracy $\simeq 40$ times better than 'allowed' by the momentum-space uncertainty relation. Of course this uncertainty relation does indeed limit the precision of any attempt to *simultaneously measure* a pair of non-commuting observables. However, as the counter example given above shows, it does not apply to *a priori* knowledge from state preparation, as used by EPR in the discussion of their gedanken experiment. There is therefore nothing remarkable (certainly no 'paradox') in the fact that non-commuting observables can be 'known' more accurately than allowed by the uncertainty relation if information about state preparation is also included, as is the case for the EPR gedanken experiment. In fact, information from state preparation is essential for the EPR analysis of (hypothetical) measurements of the system described by the wavefunction (5). According to the latter the value of x_2 is fixed by a putative measurement of x_1 and *vice versa*. In both cases the information on the unmeasured variable is given by prior knowledge of the prepared wavefunction of the system.

It has been stressed above, that no meaningful conclusions can be drawn from any gedanken experiment based upon non square-integrable wave functions. A similar criticism was made by Johansen [7] concerning a paper of Bell [8] where the erroneous conclusion was drawn, by the use of a non square-integrable wave function, that states with a positive Wigner distribution (as is in fact the case for the EPR wave function (5)) necessarily yield a local hidden variable model. A corollary is given by the 'complementary' limits discussed by Bohr [2], where an aspect of classical physics is recovered, yielding a precise position or momentum for a particle. Such exact limits are of limited physical interest since the corresponding wavefunctions are not square integrable for the conjugate variable, and so can have no physical interpretation within quantum mechanics. The Dirac δ-function is a calculational device of extreme utility. It should never be forgotten, however, that it is only a mathematical idealisation never realised in the wavefunction of any actual physical system.

In conclusion, some brief remarks are made on some widely-held conceptions concerning the meaning of the EPR paper, in the light of the above considerations.

<u>EPR showed that quantum mechanics is 'incomplete'</u>

They did not. Correcting their logical errors and replacing '$TRUE$' on the right side of the proposition (4) by '$FALSE$' it might be concluded that it was shown that quantum mechanics is 'complete'. However, as explained above, the gedanken experiment cannot be performed in the real world, essentially because of quantum mechanical uncertainty. 'Completeness' or 'incompleteness' according to EPR's definition cannot therefore be established by consideration of any gedanken experiment, no matter how idealised.

<u>The 'EPR Paradox'</u>

Because EPR's interpretation of their gedanken experiment suggests that, since both the position and momentum of a particle (albeit measured in different experiments) may both be exactly known, there must be a contradic-

tion with the momentum-space uncertainty relation. Since the analysis of the gedanken experiment is based only on quantum mechanics the latter must then be self-contradictory and therefore wrong. This erroneous conclusion arises from misinterpretation of uncertainty relations when *a priori* information is derived from state preparation.

'Spooky action-at-a-distance' in quantum mechanics

This problem results from the attempt to interpret the world governed by the laws of quantum mechanics in terms of classical concepts. The result of a quantum measurement at point A is no more 'caused' by another one at a spatially separated point B than the measurement at B is 'caused' by the one at A. Quantum mechanics predicts, however, that measurements at such causally-disconnected points may be correlated. Because of the principle of amplitude superposition in space-time[h] (a subject not discussed in the EPR paper) quantum mechanical predictions are fundamentally non-local. That this application of classical causal concepts was stressed in the EPR paper by the introduction of hypothetical 'non-interacting' sub-systems is strongly related to Einstein's known and deep philosophical prejudice[i] concerning locality (the absence of action-at-a-distance) in physics [9]. This prejudice is at variance with recent theoretical [10] and experimental [11] indications for the non-retarded nature of electrodynamical force fields. For further discussion of this point see Ref. [12]. Almost at the end of Ref. [1] occurs the following crucial passage:

'... We are thus forced to conclude that the quantum mecanical description of physical reality given by wave functions is not complete.

One could object to this conclusion on the grounds that our criterion of reality is not sufficiently restrictive. Indeed, one would not arrive at our conclusion if one insisted that two or more physical quantities can be regarded as elements of reality *only when they can be simultaneously measured or predicted.* On this point of view, since either one or the other, but not both simultaneously, of the quantities P and Q can be predicted, they are not simultaneously real. This makes the reality of P and Q depend upon the process of measurement carried out on the first system which, does not disturb the second system in any way. No reasonable definition of reality could be expected to permit this.'

(Italics in the original)

Notice the hypothetical causally disconnected subsystems that are invoked in the underlined sentence. Such subsystems do not occur in the quantum mechanical description of nature.

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[h] As, for example, in the Young double-slit experiment.

[i] See particularly the passage from a letter from Einstein to Born in 1949 cited in Ref. [9]. According to the author of the latter, the original source of Einstein's locality precept was Schopenhauer's *principium individuationis*.

both of them for stressing that EPR's reasoning was presented as a *reductio ad absurdum* argument.

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